In order to demonstrate the system capability a dc measurement was made on the cadmium resonance line with the source placed between the pole pieces of a 2-kG Alnico permanent magnet. A hole was bored through the magnet that permitted longitudinal field observations. The slope of the line profile, normalized as in Table II, is 2.1. The combination of steep slope, and pen recorder output to average over residual noise, gave an easily observed 2% modulation when the quarter-wave plate was manually rotated. Again, opposite polarities were indicated on the opposite line wings. The limitation in accuracy here was a modulation of equivalent magnitude in the absence of the magnet, resulting from slight image displacements when the quarter-wave plate was rotated, which was subtracted out for the field measurement.

CONCLUSION

The Zeeman effect technique has been successfully applied to measuring transient plasma magnetic fields within rather large uncertainty limits. It has been established that a nearly completely diamagnetic plasma is produced in the Scylla theta pinch device. The basic limitation on the method is the scarcity of spectral lines at wavelengths above 2000 Å which have sufficiently high excitation potentials to be emitted from the hot central region.

ACKNOWLEDGMENTS

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Experimental Test for the Dynamo Theory of Earth and Stellar Magnetism*

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The dynamo theories to explain earth and stellar magnetism are confronted with gerat theoretical difficulties. For this reason it has not been possible to reach a final conclusion on the feasibility of self-sustaining hydromagnetic dynamos. In order to solve these difficulties, it is suggested to test the dynamo theory experimentally under laboratory conditions. It is proposed to put a liquid conductor into a container of a rapidly rotating ultracentrifuge. To "drive" the dynamo, forceful fluid motions must be induced in the liquid conductor. This can be done either by externally applied forces, for instance by propellers, or by thermal convection. By assuming the validity of similarity laws it is possible to show that conditions presumably present in stellar-size dynamos can be simulated under laboratory conditions.

INTRODUCTION

It is still an unsolved question how magnetic fields can arise in the earth and stars.

Permanent magnetic fields to explain earth or stellar magnetism must be excluded because of the high temperatures involved. Thermoelectric effects have been considered for the earth but are certainly in stars completely insufficient to give anything comparable with the observation. On the other hand, it is a well-known fact[1] that the time for the Ohmic dissipation of a stellar magnetic field is of the order of 10^{11} yr, assuming a uniform star possessing a magnetic field being dipole-like outside the star. As a consequence of this long time scale, the origin of the stellar magnetic field must be seeked at the birth of the star or at an even earlier time.

On the other hand, it is very likely that turbulent convection within a star will eventually destroy a magnetic field in a much shorter time than 10^{11} yr. If this is the case, some kind of regenerative mechanism is necessary to explain stellar magnetic fields.

The only reasonable answer to this problem seems to be the hydromagnetic dynamo.[2] Different models for such dynamos have been proposed but no final conclusion has been reached. For an account of these efforts, together with the references, see for instance Cowling.[3]

1. FORMULATION OF THE PROBLEM

We start with the well-known equations of magneto-fluid dynamics.[4]

1. Navier-Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\text{grad})\mathbf{v}=-\frac{1}{\rho}\,\text{grad}p-\frac{1}{4\pi\rho}\mathbf{H}\times\text{curl}\mathbf{H}+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}(\zeta+\tfrac{1}{3}\eta)\text{grad div}\mathbf{v}+\frac{\mathbf{f}}{\rho}. \quad (1.1)$$

* Supported in part by the National Aeronautics and Space Administration.

[1] W. M. Elsaesser, Phys. Rev., **69**, 106 (1946).

[2] J. Larmor, Brit. Assoc. Advance Sci. Rept. **1919**, 159 (1919).

[3] T. G. Cowling, *Magnetohydrodynamics* (Interscience Publishers, Inc., New York, 1957).

[4] For instance, L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Inc., New York, 1960), p. 213 ff.

2. Generalized Ohm's law:

$$\frac{\partial \mathbf{H}}{\partial t}=\operatorname{curl}(\mathbf{v}\times\mathbf{H})+\frac{c^2}{4\pi\sigma}\nabla^2\mathbf{H}. \tag{1.2}$$

3. The energy equation:

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\operatorname{grad}s\right)=\sigma_{ik}\frac{\partial v_i}{\partial x_k}+\operatorname{div}(\lambda\operatorname{grad}T)+\frac{c^2}{16\pi^2\sigma}(\operatorname{curl}\mathbf{H})^2+q. \tag{1.3}$$

In (1.3) σ_{ik} is the viscous stress tensor:

$$\sigma_{ik}=\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_e}{\partial x_e}\right)+\zeta\delta_{ik}\frac{\partial v_e}{\partial x_e}. \tag{1.3a}$$

Summation is carried out over dummy indices.

4. The equation of continuity:

$$\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{v})=0. \tag{1.4}$$

5. The divergence equation for the magnetic field:

$$\operatorname{div}\mathbf{H}=0. \tag{1.5}$$

6. The equation of state:

$$p=p(\rho,T). \tag{1.6}$$

These equations must be supplemented by the boundary conditions on the surface of a star and of the earth's core, respectively.

In Eqs. (1.1)–(1.6) are: $\mathbf{v}$, fluid velocity; $\mathbf{H}$, magnetic field; ρ, fluid density; $\mathbf{p}$, pressure; η, ζ, the two dynamical viscosity coefficients; $\mathbf{f}$, external force density; σ, electrical conductivity; λ, heat conductivity; T, absolute temperature; s, entropy per unit mass; q, heat-source (sink) density, c, velocity of light. The system of Eqs. (1.1)–(1.6) forms a complete set of equations for magnetofluid dynamics.

For a steady-state dynamo we have to put everywhere in (1.1)–(1.4) $\partial/\partial t=0$. If $\partial/\partial t>0$, then the dynamo builds up; if $\partial/\partial t<0$, it decays. The equations can be simplified for an incompressible fluid. If we assume, in addition, that the heat conductivity λ is constant, we obtain as the simplified set of equations

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\operatorname{grad})\mathbf{v}=-\frac{1}{\rho}\operatorname{grad}p-\frac{1}{4\pi\rho}\mathbf{H}\times\operatorname{curl}\mathbf{H}+\nu\nabla^2\mathbf{v}+\frac{\mathbf{f}}{\rho}, \tag{1.7}$$

$$\frac{\partial\mathbf{H}}{\partial t}=\operatorname{curl}(\mathbf{v}\times\mathbf{H})+\frac{c^2}{4\pi\sigma}\nabla^2\mathbf{H}, \tag{1.8}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+\mathbf{v}\cdot\operatorname{grad}T\right)=\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_i}{\partial x_i}\right)\frac{\partial v_i}{\partial x_k}+\eta\nabla^2T+\frac{c^2}{16\pi^2\sigma}(\operatorname{curl}\mathbf{H})^2+q, \tag{1.9}$$

$$(\mathbf{v}\cdot\operatorname{grad}\rho)=0, \tag{1.10}$$

$$\operatorname{div}\mathbf{H}=0, \tag{1.11}$$

$$\rho=\rho(T). \tag{1.12}$$

For the energy equation, the following relation has been used:

$$Tds=c_pdT; \tag{1.13}$$

ν is the kinematic viscosity.

The force density f for the earth or a star is given by the gravitational Coriolis and centrifugal force ($\mathbf{\Omega}$ vector of rotation):

$$\mathbf{f}/\rho=\mathbf{g}_0+2\mathbf{v}\times\mathbf{\Omega}+[\mathbf{\Omega}\times\mathbf{r}]\times\mathbf{\Omega}. \tag{1.14}$$

The heat sources and heat sinks are determined by q. The heat sources in a star are determined by thermonuclear energy release. In the earth a possible heat source is the decay of radioactive material. Release of gravitational energy by still-progressing liquifaction of the core may be another source. The heat sinks are determined by the energy loss through radiation on the surface of the star and of the earth, respectively.

For the earth and the stars the fluid motion is presumably thermal convection; the dynamo is, thus, driven by heat sources. In a laboratory experiment, however, it is also possible to drive the dynamo by external forces. This can be done by pressure- or viscous-shear-flow-induced motion. Examples of pressure-induced fluid motions are those caused by pistons or propellers.

Viscous-shear-flow-induced motion occurs, for instance, in a viscous fluid enclosed between two coaxial cylinders possessing a differential rotation.

An alternative to thermal convection is of experimental advantage. Thermal convection of a magnitude required for the experiment can be accomplished only by large thermal energy release.

For the earth or a star the foregoing set of equations must be solved for given heat source distributions and boundary conditions.

The equations are nonlinear in $\mathbf{v}$ and $\mathbf{H}$. The problem is, therefore, connected with great mathematical difficulties and it has not been possible to find a solution. The question whether a hydromagnetic dynamo is possible or not is, therefore, still open. To obtain some answer in spite of these difficulties, a "restricted" dynamo problem[5–8] has been treated. In the "restricted" dynamo problem, only Eq. (1.8) is considered. After putting $\partial/\partial t=0$, this equation is

$$\operatorname{curl}(\mathbf{v}\times\mathbf{H})=(c^2/4\pi\sigma)\operatorname{curl}\operatorname{curl}\mathbf{H}. \tag{1.15}$$

[5] W. M. Elsaesser, Phys. Rev. **69**, 106 (1946); **70**, 202 (1946); **72**, 821 (1947); **79**, 183 (1950).
[6] E. C. Bullard, Proc. Roy. Soc. (London) **A197**, 433 (1950), and together with H. Gellman, Phil. Trans. Roy. Soc. (London) **247**, 213 (1954).
[7] E. N. Parker, University of Utah Reports on Earth's Magnetism and Magnetohydrodynamics, No. 7 (1954).
[8] G. Backus, Ann. Phys. (N. Y.) 4, 372 (1958).

Equation (1.15)—for a given velocity field **v**—represents an eigenvalue problem for the vector eigenfunction **H**. The eigenvalue must be equal to $c^2/4\pi\sigma$. The assumed velocity field is chosen in such a way as to follow some "reasonable" convective pattern. It has been shown that indeed solutions of (1.15) exist for such reasonable convective flow patterns. It has, however, not been demonstrated whether these solutions are compatible with the remaining equations.

2. DESCRIPTION OF THE EXPERIMENT

Instead of solving the dynamo problem theoretically, it is suggested to design laboratory experiments, in which the dynamo effect—if it exists—should be exhibited as a consequence of the solutions of magneto-fluid dynamics.

By assuming the validity of similarity laws, which are explained in the next section, the conditions of stellar size dynamos can be obtained by rapid rotation attainable in ultracentrifuges.

It is proposed to fill a spherical shell, which is part of the rotor of a centrifuge, with a liquid metal. Because of its high conductivity it is intended to use Na−K eutectic which is liquid at room temperature.

According to the dynamo theory, strong Coriolis forces seem to be important. A rapid rotation will provide the Coriolis force. A rapid rotation alone, however, is not sufficient; it will result in a rigid rotation of the fluid. For rigid rotation no dynamo effect can be expected. In the earth and stars, the rigid rotation is upset by thermal convection, excited by internal heat sources. Such heat sources can be brought into the centrifuge, for instance in the form of U^{235}, by exposing the whole apparatus to a strong neutron flux from a nuclear reactor.

Heat transmission by radiation from external sources may be an alternative. The heat will flow through the rapidly rotating apparatus and must be removed by a coolant. If the centrifuge is of the turbine type, the heat may be removed by the gas jet which drives the turbine. The convection is directed antiparallel to the direction of the centrifugal force. The heat sources for this reason are most properly placed in an equatorial belt of the spherical rotor.

One possible proposal of the apparatus is shown in Fig. 1. The spherical container is filled with liquid

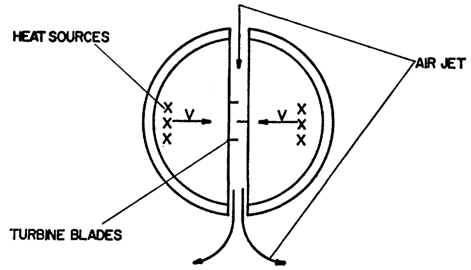

FIG. 1. Hydromagnetic dynamo *driven* by heat sources.

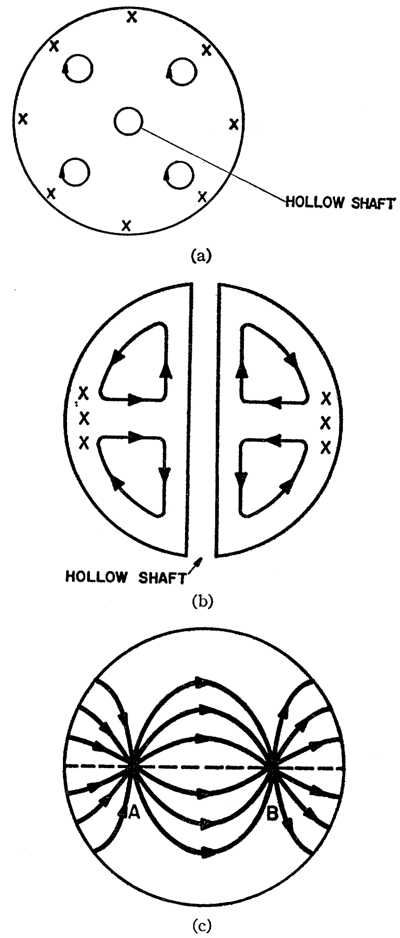

FIG. 2. Supposed shape of the convection currents for the hydromagnetic dynamo drawn in Fig. 1. × heat sources. (a) Equatorial plane; (b) meridional plane; (c) spherical surface. The latitude difference between A and B is 90°.

Na-K eutectic. The axis of the spherical container is a hollow shaft to which the container is rigidly attached. On the inner surface of the shaft are turbine blades. The centrifuge may be brought into a stable position by aerodynamic forces. The heat flows from the sources at the equatorial belt towards the shaft and is removed by the air jet which drives the centrifuge.

One might expect a convection pattern quite similar to the pattern suggested by Bullard[6] for the earth's core. Experimental work has been done by Hide[9] on the thermal convection in rotating liquids, giving some valuable in information.

We tried to draw a likely convection pattern in Fig. 2. To drive the fluid by thermal convection up to the desired velocities requires powerful heat sources. As already mentioned, an alternative to thermal convection, which will much more easily for the necessary rapid fluid motions, is viscous-shear- or pressure-induced fluid motion. For an alternative

[9] R. Hide, Phil. Trans. Roy. Soc. (London) **A249**, 441 (1957).

experiment the use of pressure forces is proposed. A pressure force can be exerted on a fluid, for instance, by moving pistons or propellers. We describe here one experimental device making use of pressure-induced fluid motion by propellers. To simulate the convective flow pattern drawn in Fig. 2, we attach to a central shaft penetrating the container two propellers, one in the "northern" and one in the "southern" hemisphere. The propeller in the "northern" hemisphere will drive the fluid towards the "north pole" and the propeller in the "southern" hemisphere will drive the fluid towards the "south pole." (See Fig. 3.)

By viscous and Ohmic dissipation, heat is generated in the liquid conductor. This heat must be removed from the dynamo by a coolant, for instance most effectively by a hydrogen gas jet. The centrifuge may be driven by a shaft, which is connected with an electric motor. The shaft is in a vertical direction and at its lower end is rigidly connected with the centrifuge. A second shaft penetrates the centrifuge from below. In contrast to the other shaft it does not participate in the rotation. The two propellers are attached to this second shaft. The propellers are thus rotating relative to a reference system rotating with the centrifuge. Since the propellers drive the fluid along the shaft in opposite directions, the resulting flow lines should be quite similar to the thermal-convection pattern described in Fig. 2.

3. SIMILARITY CONSIDERATIONS

For the design of our experiment we must possess some order-of-magnitude estimates. We have to know in which regions—accessible to laboratory experiments—dynamo effects can be expected. If one presumes that stellar magnetic fields are explained by hydromagnetic dynamos, then under conditions accessible to laboratory experiments and similar in the sense of hydromagnetic similarity, one should expect a hydromagnetic dynamo effect.

The similarity laws of magnetofluid dynamics are based on dimensionless numbers quite analogous to the case for ordinary fluid dynamics. These dimensionless numbers can be obtained directly from the equations of magnetofluid dynamics by the substitution,

$$d/dx=d/dy=d/dz=1/L. \tag{3.1}$$

L is a "size" parameter equal to a characteristic length of the flow. First we consider Eq. (1.2). The dynamo is self-sustaining if the magnetic field is constant or increasing in time.

With $\partial H/\partial t \geq 0$, there follows from (1.2) the inequality:

$$\mathrm{curl}\,\mathbf{v}\times\mathbf{H} \geq (c^2/4\pi\sigma)\nabla^2\mathbf{H}. \tag{3.2}$$

Applying (3.1) to (3.2) results in

$$4\pi\sigma Lv/c^2 \geq 1. \tag{3.3}$$

The dimensionless quantity on the left-hand side of (3.3) is the magnetic Reynolds number R_m. For the dynamo to become self-sustaining we, thus, have as a necessary condition

$$R_m \approx 1. \tag{3.4}$$

It follows from (3.3) and (3.4) that the fluid velocity necessary to excite a hydromagnetic dynamo must be at least of the order

$$v \gtrsim c^2/4\pi\sigma L. \tag{3.5}$$

For example, in the sun $\sigma \simeq 10^{16}$, $L=7\times10^{10}$ cm we need $v \gtrsim 10^{-7}$ cm/sec. In a laboratory experiment L is necessarily much smaller, and in our experiment of the order of magnitude of the radius of the centrifuge. This results in much higher fluid velocities if (3.5) is satisfied.

We propose for our experiment a liquid metal. In magnetohydrodynamics liquid mercury has been used extensively.[10–12]

With its higher electrical conductivity, which is advantageous because of relation (3.5), it is intended to use Na−K eutectic.[13]

Inserting the conductivity of Na−K eutectic into (3.5) results in

$$v > 2.2\times10^{3}\, L^{-1}(\mathrm{cm/sec}). \tag{3.6}$$

It follows that for model dynamos with diameters ranging from 1 to 10 cm, minimum fluid velocities ranging from 20 to 2 m/sec are required. In the next two sections we calculate the fluid velocity for thermal convection and pressure induction.

An order of magnitude estimate of the magnetic field strength can be obtained from the energy equation (1.9). Applying (3.1) to it and putting grad $T=\Delta T/L$, where T is a characteristic temperature difference within the dynamo, we obtain

$$\rho c_p v\frac{\Delta T}{L} \simeq \rho\nu\frac{v^2}{L^2}+\lambda\frac{\Delta T}{L^2}+\frac{c^2}{16\pi^2\sigma}\frac{H^2}{L^2}+q. \tag{3.7}$$

Solving (3.7) for H results in

$$H \simeq (4\pi\sigma^{1/2}/c)[(\rho c_p vL-\lambda)\Delta T-\rho\nu v^2-qL^2]^{1/2}. \tag{3.8}$$

TABLE I. Physical properties of Na−K eutectic (reference 13) and Hg.

	Na−K	Hg
σ	3.3×10^{16} sec^{-1}	0.94×10^{16} sec^{-1}
λ	0.26 w/cm°C	0.104 w/cm°C
c_v	0.21 cal/g°C	0.035 cal/g°C
η	2.8×10^{-3} poise	1.5×10^{-2} poise
ρ	2.6 g/cm^3	13.5 g/cm^3
α	2.76×10^{-4} °C^{-1}	1.81×10^{-4} °C^{-1}
R/R_m	2.02×10^{6}	6.8×10^{6}

[10] J. Hartmann and F. Lazarus, Kgl. Danske Videnksab. Selskab., Mat. Fys. Medd. **15**, Nos. 6 and 7 (1947).
[11] A. B. Lehnert, Arkiv Fysik **5**, 69 (1952).
[12] S. Lundquist, Phys. Rev. **83**, 307 (1951).
[13] Liquid Metals Handbook Na−K Supplement USAEC Report TID 5277, and Bureau of Ships, 1955 (unpublished).

The meaning of this equation is that the dynamo has to work as a heat engine.

We introduce into (3.8) the Reynolds number $R=Lv/\nu$ and the magnetic Reynolds number $R_m=4\pi\sigma vL/c^2$ which according to (3.4) can be put equal to one. The result is

$$H\simeq(4\pi)^{1/2}\left[\left(\rho c_p-\frac{\lambda}{vL}\right)\Delta T-\frac{\rho v^2}{R}-\frac{qL}{v}\right]^{1/2}. \quad (3.9)$$

If we neglect dissipation by thermal conduction, we get from (3.9)

$$H\simeq[4\pi(\rho c_p\Delta T-\rho v^2/R-qL/v)]^{1/2}. \quad (3.10)$$

Equation (3.10) can be brought into a dimensionless form introducing the Alfvén-Mach number,

$$M_A=H/v(4\pi\rho)^{1/2}. \quad (3.11)$$

With this definition, (3.10) can be written as

$$M_A\simeq(c_p\Delta T/v^2-1/R-qL/\rho v^3)^{1/2}. \quad (3.12)$$

If the dynamo is driven by heat sources—exciting thermal convection—and if it is under steady-state conditions, the total amount of heat supplied to the system must be equal to the total amount of heat removed from the system. The volume integral over q must, under these conditions, vanish. In Eq. (3.12) we, thus, put $q=0$ because it must be considered as an average. We then have

$$M_A\simeq(c_p\Delta T/v^2-1/R)^{1/2}. \quad (3.13)$$

To generate a magnetic field, $M_A{}^2>0$; this limits the temperature difference ΔT,

$$\Delta T>v^2/c_pR. \quad (3.14)$$

Taking for instance a fluid velocity of $v\sim10^3$ cm/sec, $c_p\simeq c_v=10^7$ ergs/g°C it follows from (3.14) that $\Delta T>10^{-7}$ °C. But for instance with $\Delta T\sim10^{-2}$ °C we get $M_A{}^2\simeq10^{-1}$; therefore, $H\simeq10^3$ G.

If the dynamo is driven by external pressure forces, q is different from zero because the heat generated by Ohmic and viscous dissipation must be removed from the system. q is a heat sink and the value of q is negative. Under steady-state conditions, the amount of heat dissipated in the system must be equal to the amount of heat removed from the system. The heat flux of the dissipated energy $\rho c_p v\Delta T$ must be equated to the negative value of the heat sink flux qL; thus,

$$-qL\simeq\rho c_p v\Delta T. \quad (3.15)$$

The result of this upon (3.12) is

$$M_A\simeq\left(2\frac{c_p\Delta T}{v^2}-\frac{1}{R}\right)^{1/2}. \quad (3.16)$$

As an order-of-magnitude estimate this is the same result as (3.13).

Apart from a factor $\frac{1}{2}$, ΔT is again limited by the inequality (3.14). If $\Delta T\gg v^2/2c_pR$, then we have approximately

$$M_A{}^2\simeq c_p\Delta T/v^2$$

and

$$H^2\simeq4\pi\rho c_p\Delta T. \quad (3.17)$$

Whether the fluid is turbulent or laminar is determined by the Reynolds number in ordinary fluid dynamics. In magnetofluid dynamics it is determined by the condition,[14]

$$R/M<2\times10^2. \quad (3.18)$$

In (3.18), R is the Reynolds number, $R=Lv/\nu$, and M is the Hartmann number:

$$M=(HL/c)(\sigma/\eta)^{1/2}. \quad (3.19)$$

We note that

$$R_M/R=4\pi\sigma\nu/c^2. \quad (3.20)$$

If the dynamo is self-sustaining, then $R_M=1$. Therefore, it follows from (3.20) that

$$R/R_M=R=c^2/4\pi\sigma\nu. \quad (3.21)$$

Equation (3.21) expresses the remarkable statement that the Reynolds number for a self-sustaining hydromagnetic dynamo depends only on the conductivity and the viscosity of the fluid.

From (3.11), (3.19), and (3.21) we get

$$R/M=R^{1/2}/M_A. \quad (3.22)$$

The magnetic energy cannot become larger than the kinetic fluid energy because the magnetic Reynolds number is much smaller than the ordinary Reynolds number. It follows that $M_A\ll1$ and, therefore,

$$R/M>R^{1/2}\sim10^3. \quad (3.23)$$

According to Murgatroyd's turbulence criterion this means the fluid is turbulent.

4. THE THERMAL CONVECTION VELOCITY

We first assume that the fluid is incompressible but that according to (1.12) $\rho=\rho(T)$. Effects arising from a finite compressibility will be discussed.

We can estimate the convection velocity then in the following way. Take a volume element of small size within the liquid. Displace the liquid in this volume element in a direction opposite to the centrifugal force g, viewed from a coordinate system at rest with the rotating centrifuge. If the thermal expansion of the liquid is positive and if the temperature gradient is directed parallel to g, then this fluid element will be accelerated in a direction antiparallel to g.

We use the nomenclature of Fig. 4. The quantities of the fluid element are given an asterisk. The fluid element in its original position characterized by index 1 is in thermal equilibrium with the surrounding fluid.

[14] W. Murgatroyd, Phil. Mat. 44, 1348 (1955).

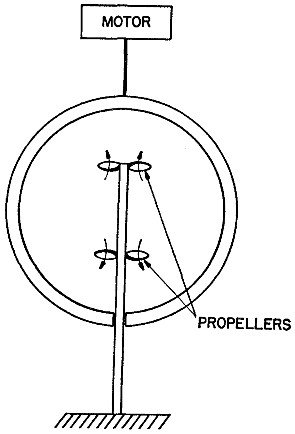

FIG. 3. Hydromagnetic dynamo *driven* by pressure-induced flow. The flow lines are presumably the same as in Fig. 2(b).

Thus,

$$\rho_1^* = \rho_1, \quad T_1^* = T_1. \tag{4.1}$$

If the fluid element is displaced by a distance dr as indicated in Fig. 4, it will possess a density and temperature equal to ρ_2^*, T_2^*.

Now if we neglect heat conductivity, then it is obvious that

$$\rho_2^* = \rho_1^* = \rho_1, \quad T_2^* = T_1^* = T_1. \tag{4.2}$$

We expand the density ρ_2 in a Taylor series,

$$\rho_2 = \rho_1 + (\partial\rho/\partial r)_1 dr. \tag{4.3}$$

Introducing the thermal expansion coefficient

$$\alpha = -(1/\rho)(\partial\rho/\partial T), \tag{4.4}$$

we can write for (4.3)

$$\begin{aligned} \rho_2 &= \rho_1 + (\partial\rho/\partial T)_1(\partial T/\partial r)_1 dr \\ &= \rho_1 - \alpha\rho_1 |\nabla T| dr. \end{aligned} \tag{4.5}$$

Putting $d\rho = \rho_2 - \rho_1$ and dropping the index 1, we obtain from (4.5)

$$d\rho = -\alpha\rho |\nabla T| dr. \tag{4.6}$$

$d\rho$ is the difference between the density of the environment of the fluid element and the fluid element. If α is positive—which is mostly the case—then the displaced fluid element is lighter than its environment. It will have a buoyancy of the magnitude

$$\begin{aligned} b &= d\rho\, g, \\ &= \alpha\rho |\nabla T| g dr. \end{aligned} \tag{4.7}$$

If the fluid is turbulent, the mean displacement of the fluid element is given by the turbulent mixing length l, which experimentally is known to be always of the same order of magnitude as the size of the boundary; thus, we may put

$$l \approx L. \tag{4.8}$$

The buoyancy acts then from $dr = 0$ to $dr = L$.

The average value of it is given by

$$\bar{b} = \tfrac{1}{2}\alpha\rho |\nabla T| L. \tag{4.9}$$

To calculate from this the convection velocity, we equate the change in kinetic energy of the fluid element $\frac{1}{2}\rho v^2$ with the gain in potential energy $\bar{b}L = \frac{1}{2}\alpha\rho |\nabla T| L^2$; the result is

$$v = L(\alpha |\nabla T| g)^{1/2}. \tag{4.10}$$

Introducing the temperature difference $\Delta T \simeq L|\nabla T|$ into (4.10) results in

$$v = (gL\alpha\Delta T)^{1/2}. \tag{4.11}$$

The value of the centrifugal force g in the centrifuge is of the order

$$g \approx V^2/L. \tag{4.12}$$

V is the maximum tangential velocity of the centrifuge. By substituting this into (4.11) we get finally

$$v \simeq V(\alpha\Delta T)^{1/2} \tag{4.13a}$$

and

$$\Delta T \simeq (1/\alpha)(v/V)^2, \tag{4.13b}$$

Take for instance $v \sim 10^3$ cm/sec, $V = 10^5$ cm/sec, and $\alpha \sim 10^{-4}$; it follows that $\Delta T \sim 1$°C.

To see how these results change under a finite compressibility, consider Fig. 5. The calculation is quite analogous to the corresponding astrophysical calculation for an ideal gas.[15] In contrast to the incompressible case, the fluid element now will expand if displaced by dr.

It is then obvious that

$$\rho_1^* = \rho_1, \quad p_2^* = p_1. \tag{4.14}$$

We expand the density in the fluid and in the fluid element at position 2 into a Taylor series:

$$\begin{aligned} \rho_2^* &= \rho_1^* + (\partial\rho_1^*/\partial r)dr = \rho_1 + (\partial\rho^*/\partial r)dr, \\ \rho_2 &= \rho_1 + (\partial\rho_1/\partial r)dr = \rho_1 + (\partial\rho/\partial r)dr. \end{aligned} \tag{4.15}$$

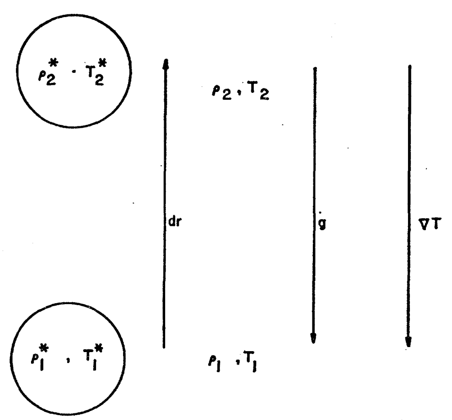

FIG. 4. To the calculation of thermal convection in an incompressible fluid.

[15] For instance: M. Schwarzchild, *Structure and Evolution of the Stars* (Princeton University Press, Princeton, New Jersey, 1958) p. 44.

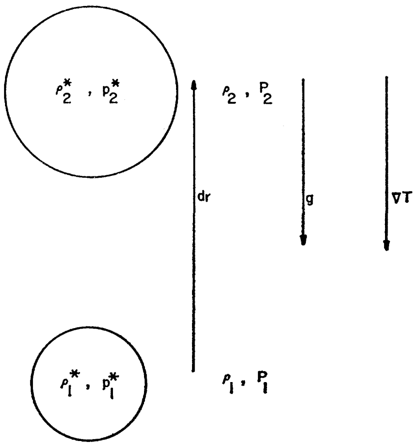

FIG. 5. To the calculation of thermal convection in a compressible fluid.

As in the previous incompressible case, heat conduction is neglected. The fluid element can, therefore, undergo an adiabatic change only. The first of the two Eqs. (4.15) gives a change in density which we shall call adiabatic density change:

$$(d\rho)_{\mathrm{ad}}=\rho_2{}^*-\rho_1{}^*=(\partial\rho/\partial r)_{\mathrm{ad}}dr. \tag{4.16}$$

The second Eq. (4.15) is the change in the density of the environment

$$\begin{aligned}d\rho&=\rho_2-\rho_1\\&=(\partial\rho/\partial r)dr.\end{aligned} \tag{4.17}$$

The equation of state is given by the three coefficients of thermal expansion α, compressibility κ, and pressure coefficient β;

$$\begin{aligned}\alpha&=-(1/\rho)(\partial\rho/\partial T)_p,\\ \kappa&=(1/\rho)(\partial\rho/\partial p)_T,\\ \beta&=(1/p)(\partial p/\partial T)_\rho.\end{aligned} \tag{4.18}$$

Between these three coefficients there exists the well-known relation

$$p\beta=\alpha/\kappa. \tag{4.19}$$

The second law of thermodynamics is written in the form

$$ds=(1/T)[du+pd(1/\rho)]. \tag{4.20}$$

In (4.20), u is the internal energy. By well-known thermodynamic relations this can be written as

$$ds=c_v\frac{dT}{T}+\left(\frac{\partial p}{\partial T}\right)_\rho d\left(\frac{1}{\rho}\right), \tag{4.21}$$

or, with the help of (4.18),

$$ds=c_v\frac{dT}{T}+\frac{\alpha}{\kappa}d\left(\frac{1}{\rho}\right). \tag{4.22}$$

Further, we write

$$d\left(\frac{1}{\rho}\right)=\left(\frac{\partial\rho^{-1}}{\partial T}\right)_p dT+\left(\frac{\partial\rho^{-1}}{\partial p}\right)_T dp=-\frac{1}{\rho}(\alpha dT-\kappa dp). \tag{4.23}$$

Eliminating dT from (4.22) and (4.23) leads to

$$ds=\left(\frac{c_v\rho}{\alpha T}\pm\frac{\alpha}{\kappa}\right)d\left(\frac{1}{\rho}\right)+\frac{c_v\kappa}{\alpha T}dp. \tag{4.24}$$

We make use of the well-known thermodynamic formula

$$c_p-c_v=T(\partial p/\partial T)_\rho(\partial\rho^{-1}/\partial T)_p, \tag{4.25}$$

or, using (4.18),

$$c_p-c_v=Tp\beta\alpha/\rho. \tag{4.26}$$

Combining this with (4.19) leads to

$$\alpha/\kappa=\rho(c_p-c_v)/\alpha T. \tag{4.27}$$

With the help of (4.27), Eq. (4.24) can be simplified. The result is

$$ds=\frac{1}{\alpha T}\left[\rho c_p d\left(\frac{1}{\rho}\right)+c_v\kappa dp\right]. \tag{4.28}$$

If the fluid element undergoes an adiabatic change, then $ds=0$. From this it follows that

$$(d\rho)_{\mathrm{ad}}=\rho\kappa(c_v/c_p)dp. \tag{4.29}$$

Therefore,

$$\left(\frac{d\rho}{dr}\right)_{\mathrm{ad}}=\kappa\rho\frac{c_v}{c_p}\frac{dp}{dr}. \tag{4.30}$$

To calculate the density change in the environment of the fluid element we use Eq. (4.23). From this equation it follows that

$$\frac{d\rho}{dr}=-\alpha\rho\frac{dT}{dr}+\kappa\rho\frac{dp}{dr}. \tag{4.31}$$

The excess in the density change causing a buoyancy is given by the difference

$$d\bar{\rho}=\frac{d\rho}{dr}dr-\left(\frac{d\rho}{dr}\right)_{\mathrm{ad}}dr=\rho\alpha\left[\frac{dT}{dr}-\frac{\kappa}{\alpha}\left(1-\frac{1}{\gamma}\right)\frac{dp}{dr}\right]. \tag{4.32}$$

In (4.32), γ is the ratio c_p/c_v.

It can be easily verified, by eliminating dρ from (4.22) and (4.23) and using (4.27), that the second term in the bracket of (4.32) is the adiabatic temperature gradient. We, thus, have

$$\left(\frac{dT}{dr}\right)_{\mathrm{ad}}=\frac{\kappa}{\alpha}\left(1-\frac{1}{\gamma}\right)\frac{dp}{dr}. \tag{4.33}$$

The influence of the compressibility on the value for the convection velocity can be included by substituting everywhere for the temperature gradient the excess of the actual temperature gradient over the adiabatic temperature gradient.

With the help of (4.27) we can write instead of (4.33)

$$\left(\frac{dT}{dr}\right)_{\text{ad}}=\frac{\alpha T}{\rho c_p}\frac{dp}{dr}. \tag{4.34}$$

We estimate the magnitude of this adiabatic temperature gradient. We put $dp/dr\sim\rho V^2/L$, where V is the tangential velocity and L the size of the centrifuge. It follows that

$$(\Delta T)_{\text{ad}}\sim\alpha TV^2/c_p\sim\alpha TV^2/c_v. \tag{4.35}$$

Taking the α value for Na–K eutectic at $T\simeq4\times10^2$ °C and assuming a value for V of the order of 10^5 cm/sec, we obtain from (4.35) a value

$$(\Delta T)_{\text{ad}}\sim0.1°\text{C}.$$

The adiabatic temperature gradient is roughly one order of magnitude smaller than the actual temperature difference of the order $\Delta T\sim1$°C calculated above. Effects caused by compressibility can be assumed to be small as long as

$$\Delta T\geq0.1°\text{C}.$$

The temperature difference must be sustained by a heat transfer to some coolant, in our case to the jet which drives the turbine of the centrifuge. If the fluid is incompressible, the heat flux Q transmitted to the jet is approximately given by

$$Q\simeq\rho c_p v\Delta T. \tag{4.36}$$

If the fluid is compressible, one has to take instead of ΔT in (4.37) the excess of the actual temperature gradient over the adiabatic temperature gradient:

$$Q\simeq\rho c_p v(\Delta T-\Delta T_{\text{ad}}). \tag{4.37}$$

Making use of (4.35) we obtain

$$Q\approx\rho c_p v\Delta T\left(1-\frac{\alpha V^2}{c_p\Delta T/T}\right). \tag{4.38}$$

In the calculation of the convection velocity we have neglected the Coriolis forces. Chandrasekhar[16] has shown that Coroilis forces, in general, inhibit convection except for the case in which the vector of angular rotation is perpendicular to g. In our situation g is the centrifugal acceleration and, therefore, perpendicular to the vector of rotation.

5. ESTIMATE OF A PROPELLER-INDUCED FLUID MOTION

We consider the fluid in a coordinate system at rest relative to the rotating container of the centrifuge. The propellers are arranged according to Fig. 3.

[16] S. Chandrasekhar, Proc. Roy. Soc. **A217**, 306 (1953).

The fluid moves toward the propeller blades with an average velocity v. The propeller accelerates a fluid element dm to a final velocity c_0. c_0 is a constant depending on the construction of the propeller blades. The change in kinetic energy per unit time of the fluid passing the propeller is thus given by

$$\frac{dE}{dt}=\frac{1}{2}(c_0{}^2-v^2)\frac{dm}{dt}. \tag{5.1}$$

The fluid is ejected by the propeller in form of a jet with a radius r. This jet radius r is apparently the propeller radius. It is, therefore, evident that

$$dm/dt=c_0\rho\pi r^2. \tag{5.2}$$

Substituting (5.2) into (5.1) results in

$$dE/dt=\tfrac{1}{2}\pi r^2\rho c_0(c_0{}^2-v^2). \tag{5.3}$$

The viscous and Ohmic energy losses are of the order of magnitude

$$P\sim L\left(\frac{4\pi\eta}{3}v^2+\frac{c^2}{12\pi\sigma}H^2\right)=\frac{4\pi\eta Lv^2}{3}\left(1+\frac{c^2}{(4\pi)^2\sigma\eta}\frac{H^2}{v^2}\right). \tag{5.4}$$

Making use of (3.11) and of (3.21) leads to

$$P\sim\frac{4\pi\eta Lv^2}{3}(1+RM_A{}^2). \tag{5.5}$$

Equating (5.5) to (5.3) and solving for the dimensionless ratio $v^2/c_0{}^2$ leads to

$$v^2/c_0{}^2\approx\frac{1}{1+(8\nu L/3r^2c_0)(1+RM_A{}^2)}. \tag{5.6}$$

Let the centrifuge possess a tangential velocity, V. The propeller, which does not participate in the rotation of the centrifuge, has relative to the centrifuge a speed c_0 of the order of magnitude

$$c_0\approx(r/L)\text{V}. \tag{5.7}$$

The flow is turbulent; in a turbulent flow the viscous dissipation is increased. The increase of the viscous dissipation in a turbulent flow through a pipe has been studied experimentally and it has been found that the friction is increased in comparison to a laminar flow by the Blasius factor

$$F=5\times10^{-3}\,R^{3/4}\sim10^2;\quad(R\sim10^6). \tag{5.8}$$

We assume that in our situation the viscous friction is increased by the same factor. We introduce a "container" Reynolds number $R_c=LV/\nu$ and rewrite (5.6), with the result

$$v\approx\frac{r}{L}V\frac{1}{[1+3\times10^2(L/r)^3(1/R_c)(1+RM_A{}^2)]^{1/2}}. \tag{5.9}$$

Assume for instance that

$r/L \simeq 10^{-1}$; $V = 3.10^4$ cm/sec, $R \sim 10^6$; $M_A{}^2 \sim 10^{-1}$.

It follows from (5.9) that

$$v \sim 10^3 \text{ cm/sec.}$$

6. THE WORKING CONDITIONS FOR THE MODEL DYNAMO

With the results of Secs. 3, 4, and 5 the working conditions of the model dynamo can be determined. We put the conditions together.

We consider first the model dynamo "driven" by heat sources. The required velocity is obtained from equation (3.5). The temperature difference is obtained by combining Eq. (3.5) with (4.13b); the result is

$$\Delta T \geq \frac{c^4}{16\pi^2\sigma^2\alpha L^2 V^2}. \quad (6.1)$$

For the values of the Na—K eutectic,

$$v \geq 2.2 \times 10^3/L \text{ cm/sec,} \quad (6.2)$$

$$\Delta T \geq 1.75 \times 10^{10}/L^2V^2 \text{ °C.} \quad (6.3)$$

With help of (6.2) and (6.3), the total heat flux according to (4.37) (putting always $c_p \simeq c_v$) is then easily calculated:

$$Q \simeq \rho c_v v \Delta T. \quad (6.4)$$

For the Na—K eutectic,

$$Q \simeq 5 \times 10^3 \Delta T/L \text{ W/cm}^2 \simeq 8.7 \times 10^{13}/L^3V^2 \text{ W/cm}^2 \quad (6.5)$$

The heat-source density q is according to (3.15) connected with Q by $q \sim qL$; thus,

$$q \approx 8.7 \times 10^{13}/L^4V^2 \text{ W/cm}^3. \quad (6.6)$$

The magnetic field strength follows from (3.17). Inserting the value (6.3), we have

$$H \sim 2.2 \times 10^9/LV \text{ G.} \quad (6.7)$$

As an example let us take $L = 10$ cm, $V = 10^4$ cm/sec. It follows $v \simeq 10^2$ cm/sec, $\Delta T \simeq 1$°C $Q \simeq 10^3$ W/cm², $q \simeq 10^2$ W/cm³, $H \sim 10^4$ G.

For the propeller-driven dynamo, Eq. (6.2) holds unchanged. The required tangential velocity of the centrifuge can be calculated with (5.9) if v is known:

$$V \approx v(L/r)[1 + 3 \times 10^2 (L/r)^3 (1/R_c)(1 + RM_A{}^2)]^{1/2}. \quad (6.8)$$

Solving for ΔT from (3.17), we have

$$\Delta T \sim 3.4 \times 10^{-8} H^2 \text{ °C.} \quad (6.9)$$

In the pressure-driven dynamo there is no restriction on ΔT such as for the convection-driven dynamo expressed by (6.3). From (6.9) we obtain a ΔT if a magnetic field still measurable is inserted into the right-hand side of (6.9). For the Alfvén-Mach number (3.11) we obtain, after introducing the expression (6.2) for v,

$$M_A \simeq 0.8 \times 10^{-4} HL. \quad (6.10)$$

From Eq. (6.4) which holds unchanged, from (6.2) and from (6.9) we obtain for the heat flux Q

$$Q \sim 1.7 \times 10^{-4} H^2/L \text{ W/cm}^2.$$

Take, for example, $L = 10$ cm, $L/r = 10$, and $H = 1$ G. It follows that $v \simeq 10^2$ cm/sec, $M_A \simeq 10^{-3}$, $V = 10^3$ cm/sec, $\Delta T \sim 3 \times 10^{-8}$ °C, and $Q \sim 10^{-5}$ W/cm².